# The Forgotten Trigonometric Functions, or
# How Trigonometry was used in the Ancient Art of Navigation (Before GPS!)

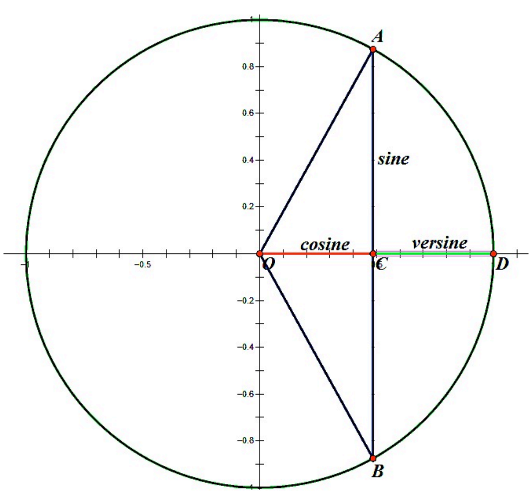
Recently, as I was exploring some mathematical concept I came across some terms that were suspiciously similar to some familiar trigonometric functions. What is an excosecant? How about a Havercosine? As it turns out, these belong to a family of trigonometric functions that once enjoyed very standard usage, in very notable and practical fields, but in recent times have become nearly obsolete.

## ***Versine***

The most basic of this family of functions is the versine, or versed sine. The versine, abbreviated versin of some angle x is actually calculated by subtracting the cosine of x from 1; that is versin(x) = 1 - cos (x). Through some manipulations with half-angle identities, we can see another equivalent expression:

$$
\begin{aligned}
versin(x) &= 1 - \cos x \\
&= \frac{2(1-\cos x)}{2} \\
&= 2\left(\sqrt{\frac{1-\cos x}{2}}\right)^2 \\
&= 2\, sin^2\left(\frac{1}{2}x\right)
\end{aligned}
$$

Notice, we do not need the $\pm$ as the versine will always represent non-negative value (since cosine ranges from negative one to positive one, one minus these values will range from zero to two). This was especially helpful as navigators used logarithms in their calculations, so they could always take the logarithm of the versine.




It is interesting that the Latin name for versine is either sinus versus (flipped sine, to distinguish from sinus rectus, the vertical sine) or sagitta, meaning arrow. It is easy to see, when looking at a drawing of the sine, cosine and versine why the reference to arrow occurs. Envision a bow and arrow; on the sketch, the arc AB is the bow, segment AB is the bowstring, and segment CD is the drawn arrow shaft, thus, the reference to an arrow. Interestingly, though it has fallen out of contemporary usage, the word sagitta is a synonym for abscissa, the x-axis.

## ***Haversine***

$$versin\ \theta = 1 - \cos\theta = 2\ sin^2\left(\frac{1}{2}\theta\right)$$

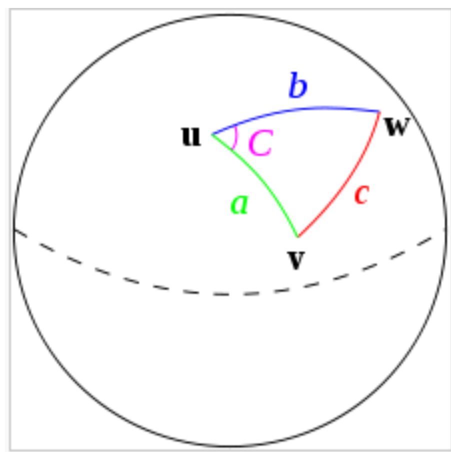
$$haversin\ \theta = \frac{1-\cos\theta}{2} = sin^2\left(\frac{1}{2}\theta\right)$$

$$cos\ \theta = 1 - versin\ \theta = 1 - 2\ haversin\ \theta$$

While this function is obviously related to the versine (literally, it is half the versine, thus haversine), it has the most practical usage associated with it.

In early times, navigators would calculate distance from their known latitudinal and longitudinal coordinates to those of their desired destination by using what became known as the Haversine formula.  Starting with the Spherical Law of Cosines, and making appropriate substitutions using the definition of haversine, and the difference of cosines identity, we can derive what is known as the Law of Haversines.  From there, we can relate it to the specific case of the earth to arrive at the Haverine formula, which then can be applied to find a distance between two points.

***Deriving the Law of Haversines***:




*The Spherical Law of Cosines*:

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

Where a, b, c are spherical arcs,
and C is the spherical angle between a and b.

First, substitute for cos c and cos C in terms of haversine:

$$1 - 2\,\text{hav}\,c = \cos a \cos b + \sin a \sin b(1 - 2\,\text{hav}\,C)$$

Then distributing on the right hand side:

$$1 - 2\,\text{hav}\,c = \cos a \cos b + \sin a \sin b - 2 \sin a \sin b\ hav\ C$$

Replacing the difference of cosines:

$$1 - 2\,\text{hav}\,c = \cos(a-b) - 2 \sin a \sin b\ hav\ C$$

Next, substitute for cos (a-b) in terms of haversin:

$$1 - 2\,\text{hav}\,c = 1 - 2\,\text{hav}(a-b) - 2 \sin a \sin b\ hav\ C$$

Last, simplify by subtracting one and dividing by -2:

$$\text{hav}\,c = \ \text{hav}(a-b) + \sin a \sin b\ hav\ C$$

***From the Law of Haversines to the Haversine Formula***:

$$\text{hav}\, c = \text{hav}(a-b) + \sin a \sin b\; hav\; C$$

Let d be the spherical distance between any two points on the surface of the earth, r be the radius of the earth, $\propto_1$ *and* $\propto_2$ be the latitudes of point 1 and point 2, respectively, and $\beta_1$ *and* $\beta_2$ be their respective longitudes. The haversine of the central angle between the points is:

$$\text{hav}\left(\frac{d}{r}\right) = \text{hav}(\propto_2 - \propto_1) + \cos\propto_1 \cos\propto_2\;\; hav\,(\beta_2 - \beta_1)$$

Since $haversin\;\theta = sin^2\left(\frac{1}{2}\theta\right)$, we have

$$sin^2\left(\frac{d}{2r}\right) = \text{hav}(\propto_2 - \propto_1) + \cos\propto_1 \cos\propto_2\;\; hav\,(\beta_2 - \beta_1)$$

And $\frac{d}{2r} = arcsin\sqrt{\text{hav}(\propto_2 - \propto_1) + \cos\propto_1 \cos\propto_2\;\; hav\,(\beta_2 - \beta_1)}$

Finally, $d = 2r\; arcsin\sqrt{sin^2\left(\frac{\propto_2 - \propto_1}{2}\right) + \cos\propto_1 \cos\propto_2\;\; sin^2\;\left(\frac{\beta_2 - \beta_1}{2}\right)}$

So, now we are able to determine the distance between two points. Letting point 1 be Atlanta, GA (33°44'56"N, 84°23'17"W) and point 2 be Sao Paulo, Brazil (23°32'0"S, 46°37'0"W), we should be able to find the distance by making appropriate substitutions and a reasonable approximation for the radius of the earth (which is not a perfect sphere). We will use the mean of the shortest and longest radii: approximately 3959 miles.

***Using the Haversine Formula***:

$$d = 2(3959)\; arcsin\sqrt{sin^2\left(\frac{-23.533° - 33.749°}{2}\right) + \cos 33.749° \cos(-23.533°)\; sin^2\;\left(\frac{46.617° - 84.388°}{2}\right)}$$

$$d = 7918\; arcsin\sqrt{sin^2(-28.641°) + \cos 33.749° \cos(-23.533°)\; sin^2\;(-18.886)}$$

$$d = 7918\; arcsin\sqrt{(-0.4793)^2 + (0.8315)(0.9168)(-0.3237)^2}$$

$$d = 7918\; arcsin\sqrt{.3096}$$

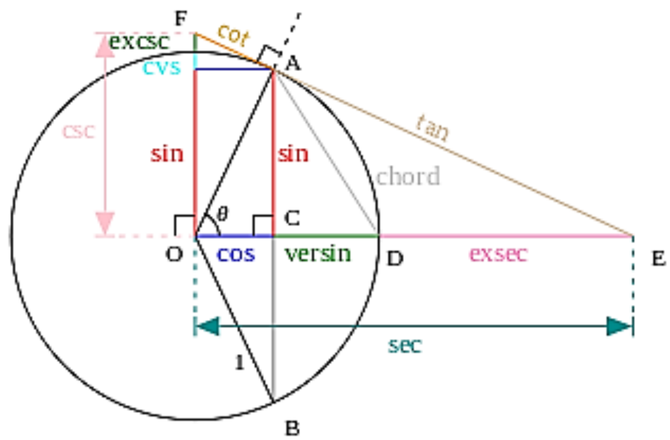
$$d = 7918\; arcsin\, 0.5564$$

$$d = 7918\;(33.8083)\left(\frac{\pi}{180}\right) \approx 4672\;\; miles$$

Check out the link on my home page for an Excel Worksheet that will calculate the distance between two locations using the Haversine Formula and their latitudes and longitudes.

There are many other "classical" trigonometric functions that are not as practical and have been abandoned in favor of manipulation of the six well-known (and loved) trig ratios.  The table and diagram below illustrate how these functions are defined.

| | |
|---|---|
| $vercosine\ \theta$ | $1 + cosine\ \theta$ |
| $coversine\ \theta$ | $1 - sine\ \theta$ |
| $covercosine\ \theta$ | $1 + sine\ \theta$ |
| $havercosine\ \theta$ | $\frac{1 + cosine\ \theta}{2}$ |
| $hacoversine\ \theta$ | $\frac{1 - sine\ \theta}{2}$ |
| $hacovercosine\ \theta$ | $\frac{1 + sine\ \theta}{2}$ |
| $exsecant\ \theta$ | $secant\ \theta - 1$ |
| $excosecant\ \theta$ | $cosecant\ \theta - 1$ |




As we evolve technologically, it is important that we remember and recall the steps we have taken to achieve the stature we have reached thus far.  It is valuable for us to remember those "giants" upon whose shoulders we happen to currently stand.

Weisstein, Eric W. "Versine." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/Versine.html

http://en.wikipedia.org/wiki/Versine

http://en.wikipedia.org/wiki/Haversine_formula

Kells, Kern & Bland.  *Plane and Spherical Trigonometry*.  York, PA:  Maple Press Company, 1935.  Print.